Computability and Computational Complexity, A.Y. 2025–2026

Written test

Thursday, September 3, 2026

Exercise 1

Let $\Sigma = \{\texttt{0}, \texttt{1}\}$ be the set of binary digits and consider the language $L \subset \Sigma^*$ that contains all and only binary strings with alternating `0` and `1` (i.e., every `0` is followed by `1` and vice versa) with at most one exception (repeated digits `00` or `11`). For instance, the following strings belong to L because there is at most one repetition (underlined):

$$\texttt{0} \quad \underline{\texttt{11}} \quad \texttt{01} \quad \underline{\texttt{00}}\texttt{1} \quad \texttt{1010101} \quad \texttt{1010}\underline{\texttt{11}}\texttt{01} \quad \underline{\texttt{00}}\texttt{10101} \quad \texttt{1}\underline{\texttt{00}}\texttt{101010} \quad \varepsilon.$$

The following don't (they contain more than one pair of adjacent equal digits, `00` or `11`):

$$\underline{\texttt{0011}} \quad \texttt{01}\underline{\texttt{00}}\texttt{10101}\underline{\texttt{00}}\texttt{1} \quad \underline{\texttt{11}}\texttt{010}\underline{\texttt{11}} \quad \texttt{010101}\underline{\texttt{00}}\texttt{101}\underline{\texttt{00}}\texttt{10101} \quad \texttt{01}\underline{\texttt{000}}\texttt{101010101}.$$

Moreover, let $L_n = L \cap \bigcup_{i=0}^{n} \Sigma^i = \{s \in L : |s| \leq n\}$ be the subset of L with strings of length n or less.

1.1) For each of the following properties of a Turing machine $\mathcal{M}$, prove whether they are computable or not, invoking Rice's Theorem if possible:

$$\begin{aligned}
\mathcal{P}_1 &= \{\mathcal{M} : \mathcal{M} \text{ decides } L\} \\
\mathcal{P}_2 &= \{\mathcal{M} : \mathcal{M} \text{ decides } L_5\} \\
\mathcal{P}_3 &= \{\mathcal{M} : \mathcal{M} \text{ decides } L \text{ in less than } 10^6 \text{ steps}\} \\
\mathcal{P}_4 &= \{\mathcal{M} : \mathcal{M} \text{ decides } L_5 \text{ in less than } 10^6 \text{ steps}\}
\end{aligned}$$

1.2) Write a single-tape, deterministic Turing machine (with alphabet $\{\llcorner\!\lrcorner, \texttt{0}, \texttt{1}\}$) that decides L.

Hint — *Be careful in $\mathcal{P}_3$ and $\mathcal{P}_4$: we require $\mathcal{M}$ to halt within 10^6 steps regardless of the input. The machine in 1.2 needs only perform a single scan, provided that it remembers two items of information: (i) the last symbol seen, and (ii) whether it has already met an exception (repeated digit) or not.*

Exercise 2

Let $L_1, L_2 \in \mathbf{NP}$, and suppose $\mathbf{P} \neq \mathbf{NP}$. Does $L_1 \cup L_2 \in \mathbf{NP}$? Does $L_1 \setminus L_2 \in \mathbf{NP}$?
Try to be as clear and formal as possible.

Exercise 3

Consider the following classical **NP**-complete languages:

$$\begin{aligned}
\text{INDEPENDENT SET} &= \{(G, k) : \text{undirected graph } G \text{ has an independent set of size } k\} \\
\text{VERTEX COVER} &= \{(G, k) : \text{undirected graph } G \text{ has a vertex cover of size } k\}.
\end{aligned}$$

Describe a polynomial-time reduction from INDEPENDENT SET to VERTEX COVER.

Hint — *Remember that, given an undirected graph $G = (V, E)$ an independent set is a subset of nodes $V' \subseteq V$ with no connections between any pair (i.e., $v_1, v_2 \in V' \Rightarrow \{v_1, v_2\} \notin E$).*
A vertex cover is a subset of nodes $V' \subseteq V$ such that every edge in E has at least one endpoint in V' (i.e., $e \in E \Rightarrow e \cap V' \neq \emptyset$).
For the reduction observe that, if $V' \subseteq V$ is a vertex cover and $v_1, v_2 \notin V'$, then the edge $\{v_1, v_2\} \notin E$ (but explain why).