Computability and Computational Complexity, A.Y. 2024–2025

Written test

Monday, February 17, 2025

Exercise 1

The following example appears in the Clay Institute webpage to motivate the inclusion the **P** vs. **NP** issue among its Millennium Prize Problems:

> *Suppose that you are organizing housing accommodations for a group of four hundred university students. Space is limited and only one hundred of the students will receive places in the dormitory. To complicate matters, the Dean has provided you with a list of pairs of incompatible students, and requested that no pair from this list appear in your final choice.*

We are interested in the general problem with N students and n dormitory places (in the example, $N = 400$ and $n = 100$); let the question be "Will you be able to fill all n dormitory places?"

1.1) Prove that the problem is in **NP**.

1.2) Knowing that 3SAT is complete, prove that the Clay Institute problem is **NP**-complete by an appropriate reduction.

Hint — *For point 1.2, observe that the Clay Institute example problem is just the rephrasing of a well known **NP**-complete problem, and imitate the reduction seen in class.*

Exercise 2

In the following property definitions, a finite alphabet Σ is given; $\mathcal{M}$ spans all DTMs on Σ; $x \in \Sigma^*$ spans all strings; finally, $\mathcal{M}(x)$ represents the computation of $\mathcal{M}$ on input x:

$$\begin{aligned}
\mathcal{P}_1 &= \{\mathcal{M} : \forall x\ \mathcal{M}(x) \text{ leaves the initial state } q_0 \text{ at the first step}\} \\
\mathcal{P}_2 &= \{\mathcal{M} : \forall x\ \mathcal{M}(x) \text{ never enters the initial state } q_0 \text{ after the first step}\} \\
\mathcal{P}_3 &= \{\mathcal{M} : \forall x\ \mathcal{M}(x) \text{ changes state at every step}\}.
\end{aligned}$$

2.1) Prove that none of the above properties is semantic.

2.2) For each of the properties above, prove whether it is computable or not.

2.3) Prove or disprove the following statement:

Every trivial property of Turing machines is semantic.

Exercise 3

3.1) State Cook-Levin's Theorem.

3.2) Outline the Theorem's proof.