Computability and Computational Complexity, A.Y. 2024–2025

Written test

Thursday, January 9, 2025

Exercise 1

1.1) Define the space complexity class **L**.

1.2) Consider the language of all strings in $\{0, 1\}^*$where the number of 1's is strictly larger than the number of zeroes, e.g.:

$$011 \in L,\ 1101 \in L,$$
$$10 \notin L \text{ (the number of 1's must be } \textit{strictly} \text{ larger)},$$
$$1 \in L,\ 11111111 \in L,$$
$$0 \notin L,\ 101010 \notin L.$$

Prove that $L \in \mathbf{L}$.

Hint — *For 1.2: start by sketching down a (pseudocode) program that decides L.*

Exercise 2

2.1) Provide a precise definition of *semantic* and *trivial* in reference to properties of Turing Machines.

2.2) State Rice's Theorem.

2.3) Why does the proof of Rice's Theorem fail if the property is *not* semantic (e.g., "the TM has more than 100 states")?

Exercise 3

In the central hall of the ancient temple, having dodged an inordinate amount of boobytraps, Indiana Jones is ready to grab the golden statue from its pedestal; however, to avoid triggering even more deadly traps on his way back, he must replace the statue with something having its *exact* weight. Alas, his bag of sand was ripped by an arrow; all he has got is a large and heterogeneous set of archaeologist's tools whose individual weights he had luckily annotated in his notebook before leaving his office. Of course, his experienced look can precisely estimate the statue's weight he is so eager to match.

3.1) Show that, given the high precision of the ancient mechanism and the diversity of tools in his belt, he might need a very long[a] time before being able to determine if there is a combination of tools whose weight matches the statue's.

3.2) An oracle is quietly sitting in a corner of the hall. He was gifted with a very peculiar ability: when presented with any intricate map of rooms connected by tunnels, the oracle is immediately able to point out a round trip that visits all rooms exactly once, provided that such path exists. Prove that Indiana Jones could exploit the oracle's ability in order to solve the problem of matching the statue's weight in a reasonable[b] time.

Hint — *Assume that all weights are known with the precision of one gram. Both questions require answers in the form of reductions from/to known difficult (i.e., **NP**-complete) problems. Here is a list of languages that we already know to be **NP**-complete:* SATISFIABILITY, 3-SATISFIABILITY, CLIQUE, INDEPENDENT SET, INTEGER LINEAR PROGRAMMING, VERTEX COVER, 3-VERTEX COLORING, *SUBSET SUM*, KNAPSACK, HAMILTONIAN PATH, DIRECTED HAMILTONIAN CYCLE, *HAMILTONIAN CYCLE*, TRAVELING SALESMAN PROBLEM.

[a] I.e., exponential with respect to the number of tools in the worst case.

[b] I.e., polynomial with respect to the number of tools in the worst case.