Computability and Computational Complexity, A.Y. 2023–2024

Written test

Monday, February 19, 2024

Exercise 1

Let Σ be a finite alphabet, and $L_{\text{R1}}, L_{\text{R2}}, L_{\text{RE1}}, L_{\text{RE2}} \subseteq \Sigma^*$ be four languages on Σ. L_{R1} and L_{R2} are recursive, while L_{RE1} and L_{RE2} are recursively enumerable, but not recursive.

1.1) For each of the following languages, state if they are recursive, recursively enumerable, or none, and motivate your answers:

- $L_1 = L_{\text{R1}} \cup L_{\text{R2}}$;
- $L_2 = L_{\text{R1}} \cap L_{\text{R2}}$;
- $L_3 = L_{\text{R1}} \cup L_{\text{RE1}}$;
- $L_4 = L_{\text{R1}} \cap L_{\text{RE1}}$;
- $L_5 = L_{\text{RE1}} \cup L_{\text{RE2}}$;
- $L_6 = L_{\text{RE1}} \cap L_{\text{RE2}}$.

1.2) State whether the following properties of Turing machines are computable or not, and motivate your statements:

- $\mathcal{P}_1 = \{\mathcal{M} :\ \mathcal{M} \text{ decides } L_{\text{R1}}\}$;
- $\mathcal{P}_2 = \{\mathcal{M} :\ \mathcal{M} \text{ decides } L_{\text{RE1}}\}$;
- $\mathcal{P}_3 = \{\mathcal{M} :\ \text{if } |x| < 100, \text{ then } \mathcal{M} \text{ decides } x \in L_{\text{R1}} \text{ in no more than } |x|^2 + 1 \text{ steps}\}$;
- $\mathcal{P}_4 = \{\mathcal{M} :\ \text{if } |x| < 100, \text{ then } \mathcal{M} \text{ decides } x \in L_{\text{RE1}} \text{ in no more than } |x|^2 + 1 \text{ steps}\}$.

Exercise 2

Let $L \in \mathbf{P}$ be a deterministic polynomial-time language on finite alphabet Σ, and let L' and L'' be defined as follows:

- $L' = \Sigma^* \times L \times \Sigma^* = \{w_1 w_2 w_3 :\ w_1, w_3 \in \Sigma^* \wedge w_2 \in L\}$, the language of all strings on alphabet Σ that contain a word from L as a substring (contiguous sequence of symbols);
- $L'' = \{\sigma_1\sigma_2\sigma_3 \dots \sigma_n \in \Sigma^* :\ \exists k, i_1, i_2, \dots i_k\, (0 \le k \le n \wedge 1 \le i_1 < i_2 < \dots < i_k \le n \wedge \sigma_{i_1}\sigma_{i_2} \dots \sigma_{i_k} \in L)\}$, the language of all strings containing a (non necessarily contiguous) subsequence of symbols that compose a word in L.

For instance, if "cat" $\in L$, then "lo<u>cat</u>ion" and "<u>cat</u>alog" belong to both L' and L'', while the words "de<u>c</u>or<u>at</u>ion" and "<u>c</u>roiss<u>a</u>n<u>t</u>" only belong to L''.

2.1) Discuss the deterministic time complexity of L' and L''.

2.2) What about their non-deterministic time complexity?

Exercise 3

State and prove Rice's theorem about the undecidability of semantic, non-trivial properties of Turing machines.