

## Written exam

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### Exercise 1

Consider the following language in  $\{0, 1\}^*$ :

$$K = \{0^n 1^n : n \in \mathbb{N}\} = \{\varepsilon, 01, 0011, 000111, 00001111, 0000011111, \dots\}$$

i.e., all strings composed by a sequence of zeroes followed by the *same* number of ones.

**1.1)** Write a single-tape Turing Machine with alphabet  $\Sigma = \{\_, 0, 1\}$  that recognizes  $K$ .

**1.2)** Prove or disprove the decidability of each of the following properties of TMs:

$$\mathcal{P}_1 = \{\mathcal{M} : \mathcal{M} \text{ decides } K\},$$

$$\mathcal{P}_2 = \{\mathcal{M} : \mathcal{M} \text{ decides } K \text{ in less than 100 steps}\},$$

$$\mathcal{P}_3 = \{\mathcal{M} : \mathcal{M} \text{ decides } K \cap \Sigma^{100} \text{ (i.e., strings in } K \text{ not longer than 100 symbols)}\}.$$

*For 1.1 use any notation you like, and encode acceptance and rejection as you prefer (0/1 on tape, two different halting states, etc.).*

### Exercise 2

Prove that  $K$ , the language defined in Exercise 1, belongs to the complexity class **L**.

*While 1.1 required a single-tape machine, class **L** has a different assumption. Here, however, you are not asked to write down the TM: just a few lines of pseudocode will do.*

### Exercise 3

Let  $L_1, L_2 \in \mathbf{NP}$ . Does  $L_1 \cup L_2 \in \mathbf{NP}$ ? Does  $L_1 \cap L_2 \in \mathbf{NP}$ ? Why?

*Be as formal as you can, e.g.: “Since  $L_1 \in \mathbf{NP}$ , then there is a TM  $\mathcal{M}_1$  such that...”*

### Exercise 4

Consider the following classical **NP**-complete languages:

**CLIQUE** =  $\{(G, k) : \text{Undirected graph } G \text{ has a completely connected subgraph of size } k\},$

**INDSET** =  $\{(G, k) : \text{Undirected graph } G \text{ has a completely disconnected subgraph of size } k\}.$

**4.1)** Describe a polynomial-time reduction from one language to the other.

**4.2)** Show that  $\mathbf{CLIQUE} \cap \mathbf{INDSET} \neq \emptyset$ .

*For 4.1, choose the direction you like. In 4.2, don't be afraid of simple answers: to show that a set is not empty, you just need to find an element in it.*