**Computability and Computational Complexity, A.Y. 2019–2020**

# Written exam

*Mauro Brunato*

Wednesday, January 15, 2020

## Exercise 1

Consider the UNIVERSITY HIRING decision problem:

> A university needs to hire the teaching staff for a new degree, for which a set $T$ of topics must be taught. The executive board received $n$ applications from prospective teachers, and every applicant $i \in \{1, \dots, n\}$ has knowledge of a subset $S_i \subseteq T$ of the required topics. The budget allows for hiring at most $k \leq n$ teachers. Is there a choice of $k$ applicants so that all teaching topics are covered?

An instance of the problem consists of parameters $n, k, T, S_1, \dots, S_n$.

**1.1)** Prove that UNIVERSITY HIRING $\in$ **NP**.

**1.2)** Prove by reduction that UNIVERSITY HIRING is complete in the class **NP**. The reduction can refer to any language discussed during the course.

**1.3)** Prove that if $k$ is kept constant (e.g., $k = 10$), then the problem's asymptotic complexity is polynomial wrt input size.

## Exercise 2

**2.1)** When is a language recursive? When is it recursively enumerable?

**2.2)** Prove that the following property $\mathcal{P}$ of Turing machines $\mathcal{M}$ is not recursive:

$$\mathcal{P} = \{\mathcal{M} : \mathcal{M}(\varepsilon) \text{ halts after an even number of steps}\}$$

where $\varepsilon$ is the empty input string.

**2.3)** Prove that $\mathcal{P}$ is recursively enumerable.

Hint — *Point 2.3 can be proved by explicitly outlining an enumeration algorithm.*

## Exercise 3

Consider the following language:

$$S = \left\{ \big(x \in \{0,1\}^*, k \in \mathbb{N}\big) : x \text{ contains a subsequence of } k \text{ adjacent 0's} \right\}.$$

For example, $(101000101, 3) \in S$ because the binary string contains 3 consecutive zeroes, while $(101000101, 4) \notin S$ because the binary string does not contain 4 consecutive zeroes.

**3.1)** Prove that $S \in$ **P**.

**3.2)** Prove that $S \in$ **L**.

Hint — *Again, both points can be proved by describing an algorithm and showing that it has the required property.*